The mean acoustic field in layered media with rough interfaces

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Abstract

An algorithm is presented for determining the mean acoustic field in a layered medium containing rough interfaces. It is assumed that scattering by the rough interfaces when considered separately and in the absence of sound speed and density variation can be well-approximated. It is also assumed that propagation in layered media with flat interfaces can be well approximated. The present work shows how these results can be combined to yield the mean field in a stack of layers with variable sound speeds and densities which are separated by rough interfaces.

I Introduction

In two previous papers [1, 2] it was shown that the mean acoustic field in a single-layered medium with statistically rough boundaries could be expressed as an acoustic field in the same layer with flat boundaries, but with boundary conditions described by effective reflection coefficients. The effective reflection coefficients were constructed from the mean scattering amplitudes for the interfaces calculated when they separate two homogeneous half-spaces, plus corrections involving fluctuations of these half-space scattering amplitudes mediated by propagation between the interfaces of the layer. In both I and II only layers with constant density and constant sound speed were considered. The effective reflection coefficients were derived using coupled up- and down-going plane-wave solutions which incorporated the boundary condition through half-space scattering amplitudes. This was one of the primary features of the treatment in I and II: the half-space solutions could be used directly in the construction of the Green function for the layer, so that whatever approximations are known for the half-space problem needn't be rederived for the layered problem. For example, non-perturbative approximations of half-space scattering amplitudes, such as the small-slope approximation of Voronovich [3], could be used. The physics of propagation which was incorporated in the solutions of I and II can be summarized by saying that effective scattering amplitudes (or reflection coefficients) must account for all processes in which a wave of given wavevector is forward scattered. In a layered medium there are processes, involving either specular reflection or scattering at two or more interfaces, which allow forward scattering, and which are not accounted for in the half-space scattering amplitudes. Mean half-space scattering amplitudes only account for scattering at a single interface.

In this work, the results of II are generalized to the case of media with sound speed profiles and densities which vary continuously in depth, and to include the description of transmission through rough interfaces. In papers I and II, although approximations for scattering at the interfaces did not need to be rederived in the layer, the construction of the Green function did need to be rederived. As a result, a pair of coupled integral equations needed to be solved to reproduce the source. Here, plane waves cannot be used because the sound speed or density may be variable. One way of incorporating plane wave information without using plane waves is to note that that plane wave solutions imply a non-local (in wavenumber) impedance boundary condition, and then to assume this non-local impedance boundary condition applies even when the media on either side of the interface do not support plane waves. This leads to a very

elgant solution when transmission through the interface need not be considered. However, when transmission is important, it is awkward to formulate a generalization of the impedance which accounts for transmission in both directions through the interface.

The following alternate proceedure is equivalent to the impedance method and is expressed directly in terms of reflection and transmission operators. Assume that a rough interface separating two possibly inhomogeneous layers can be replaced by a flat interface which has the same reflection and transmission amplitudes. This assumption is nearly the same as the Rayleigh hypothesis in that it will be further assumed that if the surrounding media in the immediate vicinty of the interface are homogeneous, then plane wave expansions of the field exists and can be continued onto the flat replacement of the true interface. A second assumption, which will be used throughout this paper, is that even in variable sound speed media, the solution of the acoustic problem can be obtained by replacing the true medium in the vicinity of the surface by one having constant sound speed and density in thin layers on either side of the interface. The constant sound speed and densities are taken to be the values of the sound speed and density in the surrounding medium at the boundary between the thin layer. See Figure 1. Sound entering the layers adjacent to the boundary, rattles back and fourth between the rough interface and the fictitious boundary between the constant sound speed region and the rest of the medium. The rest of the medium may include other interfaces and and sound speed variation which returns scattered sound to the interface in question. It is this return of energy which leads to effective reflection coefficients in a stack of layers.

Section II develops the ideas just discussed and applies methods described by Brown et al [4] to determine effective reflection and transmission coefficients for the mean field in a stack of layers. In section III contact is made with earlier work, an estimate of the size of the effects causes by the return of energy to a rough interface is discussed, and and estimate of the shift (caused by interface roughness) in modal wavenumbers in a many-layered waveguide is given. Despite the complexity of the following sections, the resulting algorithm is simple to describe: compute the field in a layered medium using flat interfaces with boundary conditions determined by mean half-space scattering amplitudes. From this field evaluated at a particular flat interface, reflection coefficients for the remainder of the medium can be found. These and the fluctuations of the reflection and transmission amplitudes at the surface in question can be used to compute effective reflection coefficients. These effective amplitudes determine effective boundary conditions at each interface from which the mean field can be re-calculated in a manner similar the way

the flat-interface field is calculated. Self-energy-like corrections to the mean reflection and transmission calculated when the interface separates homogeneous half-spaces arise because the mean field must account for all possibilities of forward scattering. For example, even though double scattering on each interface might be included in the mean half-space reflection and transmission coefficients, these half-space coefficients do not include the possibility of scattering out of the forward direction at one interface and then reflecting at another returning to the first to be scattered back into the forward direction by roughness on the original scattering surface.

II The Net Reflection Matrix

In this section a simple formula for the net reflection and transmission at an interface embedded in a stack of variable sound speed layers will be developed. The mean of the net reflection then gives effective reflection and transmission coefficients. Voronovich [5] has used similar ideas to treat a bounding surface, and the presentation here generalizes his work only in that an interior interface is considered and effective reflection and transmission coefficients for the mean field are developed explicitly.

To establish notation, first consider a rough interface S separating two homogeneous half-spaces. Denote the amplitude of a plane wave of horizontal wavevector $\mathbf{Q}$ incident on the interface from above by $\phi_1^{inc}(\mathbf{Q})$ and the amplitude of a plane wave, again with horizontal wavevector $\mathbf{Q}$, incident from below by $\phi_2^{inc}(\mathbf{Q})$. Because the interface is rough, these incident plane waves are converted into plane waves with horizontal wavenumbers $\mathbf{K}$ leaving the interface with amplitudes $\phi_1^{out}(\mathbf{K})$ above the surface and $\phi_2^{out}(\mathbf{K})$ below the surface. The relationship between the incident amplitudes and the outgoing amplitudes is given by the matrix operator of reflection and transmission coefficients, $R(\mathbf{K}, \mathbf{Q})$. If the amplitudes are combined into vectors so that

$$\phi^{inc}(\mathbf{Q}) = \begin{pmatrix} \phi_1^{inc}(\mathbf{Q}) \\ \phi_2^{inc}(\mathbf{Q}) \end{pmatrix} \tag{1}$$

and

$$\phi^{out}(\mathbf{K}) = \begin{pmatrix} \phi_1^{out}(\mathbf{K}) \\ \phi_2^{out}(\mathbf{K}) \end{pmatrix} \tag{2}$$

then

$$\phi^{out}(\mathbf{K}) = \int d\mathbf{Q} R(\mathbf{K}, \mathbf{Q}) \phi^{inc}(\mathbf{Q}), \tag{3}$$

where $R(\mathbf{K}, \mathbf{Q})$ is the matrix of reflection and transmission amplitudes associated with the rough interface separating homogeneous half-spaces:

$$R(\mathbf{K}, \mathbf{Q}) = \begin{pmatrix} R_{1,1}(\mathbf{K}, \mathbf{Q}) & T_{1,2}(\mathbf{K}, \mathbf{Q}) \\ T_{2,1}(\mathbf{K}, \mathbf{Q}) & R_{2,2}(\mathbf{K}, \mathbf{Q}) \end{pmatrix}. \tag{4}$$

The reflection amplitude $R_{1,1}$ describes scattering from the upper medium (1) back into the upper medium, $T_{1,2}$ describes scattering of plane waves incident from the lower medium (2) transmitted into the upper medium etc. The relation between incident and outgoing plane wave amplitudes with be abbreviated further by the operator equation

$$\phi^{out} = R\phi^{inc} \tag{5}$$

Now suppose S is one of many interfaces separating many layers with variable sound speeds and densities. Just above S draw an imaginary flat surface S_1 at z_1 and below S draw a surface S_2 at z_2. See Figure 1. In the thin layer above S replace the true sound speed $c(z)$ and density $\rho(z)$ by

$$c_1 = \lim_{z \to z_1^+} c(z) \tag{6}$$

$$\rho_1 = \lim_{z \to z_1^+} \rho(z). \tag{7}$$

Likewise, in the thin layer just below S replace the true sound speed and density by

$$c_2 = \lim_{z \to z_2^-} c(z) \tag{8}$$

$$\rho_2 = \lim_{z \to z_2^-} \rho(z). \tag{9}$$

An upgoing plane wave in the thin layer above S will be reflected back toward S by the surface S_1 at z_1 into this layer according to the scattering amplitude $R_{0,1}$. This scattering or reflection occurs because, although the sound speed and density are continuous at z_1, upward traveling plane waves will encounter the rest of the medium above S_1 which can return and scatter these waves toward S. Likewise the surface S_2 at z_2 has a scattering ampltude for down-going waves being returned upward given by $R_{0,2}$. If the remainder of the medium is not horizontally homogeneous, these amplitudes will not be diagaonal in horizontal wavevector.

Now consider what happens if a vector of up- and down-going waves ϕ^{inc} whose origin is somewhere else in the medium, are incident on the surface S. These hit the surface S, are

scattered according to R and are returned to S by $R_{0,1}$ and $R_{0,2}$ and scattered again. The result of all this scattering is that the orginal incident amplitudes ϕ^{inc} are converted into net amplitudes

$$\psi^{inc} = \begin{pmatrix} \psi_1^{inc} \\ \psi_2^{inc} \end{pmatrix} \tag{10}$$

according to the matrix operator equation

$$\psi^{inc} = \phi^{inc} + R_0 R \psi^{inc}. \tag{11}$$

See Figure 2. The the kernel of the matrix operator R_0 is given by

$$R_0(\mathbf{K}, \mathbf{Q}) = \begin{pmatrix} R_{0,1}(\mathbf{K}, \mathbf{Q}) & 0 \\ 0 & R_{0,2}(\mathbf{K}, \mathbf{Q}) \end{pmatrix}. \tag{12}$$

Solving this equation gives the net incident field amplitudes as

$$\psi^{inc} = \frac{1}{1 - R_0 R} \phi^{inc}. \tag{13}$$

This means that the net out-going amplitudes are given by

$$\psi^{out} = R \frac{1}{1 - R_0 R} \phi^{inc}, \tag{14}$$

and that the net reflection (matrix) operator is given by

$$R_{net} = R \frac{1}{1 - R_0 R}. \tag{15}$$

The factor $1/(1 - R_0 R)$ could also be obtained by summing the geometric series obtained by considering all possible reflections and transmissions. Note that there are no phase factors in these expressions; it is assumed that the surfaces S_1 and S_2 can be taken to be arbitrarily close to the flattened scattering surface S which is nevertheless characterized by the scattering R.

The same algebra can be performed by considering only one side of a rough interface. Then one uses the reflection amplitude R of the interface computed when on one side (say the lower side, for definiteness) there is a constant sound speed medium from which plane waves approach the interface, and when the other (upper) side contains arbitrary structure. Although R is a plane wave scattering amplitude, it contains information about the non-homogeneous structure of the medium on the far side of the interface. It can be computed in most cases by using a projection of the full 2×2 scattering matrix and knowledge of $R_{0,1}$. The arguments used

above can be repeated for variable sound speed in the lower medium by inserting a thin layer of constant sound speed near the interface. If R can be found, the 2×2 matrix operators just discussed become 1×1 operators, but formal results such as Eq.15 remain unchanged. See Fig 3.

II.1 The mean net reflection operator

The net reflection operator is random because the roughness on the surface S is random, and therefor R is random. Furthermore scattering by the remainder of the waveguide, which is characterized by R_0, is also random. In this paper, it will be assumed that R_0, is statistically independent of R. In any case, one can first try to average R_{net} conditionally on the value of R_0. Averaging R_{net} according to Eq.15 requires the average of the inverse of a random operator. The field theoretic techniques described in the appendix of Ref.[4] provide ready-made tools for this purpose. To use these tools, write Eq.15 as

$$R_{net} = \frac{1}{R_0}\left[-1 + \left(\frac{1}{R_0^{-1} - R}\right)\frac{1}{R_0}\right]. \tag{16}$$

Holding R_0 fixed, the quantity $1/(R_0^{-1} - R)$ looks like the looks like a Green function $G = 1/(G_0^{-1} - V)$. The formalism in Ref.[4] then shows that the mean of $1/(R_0^{-1} - R)$ can be written in terms of a (mean) self-energy, Σ :

$$\langle G \rangle \equiv \langle \frac{1}{R_0^{-1} - R} \rangle = \frac{1}{R_0^{-1} - \langle R \rangle - \Sigma}. \tag{17}$$

Brown et al [4] show that the self-energy Σ can be written in terms of a scattering operator T which is defined by

$$T = (\Delta R - \Sigma)\frac{1}{1 - \langle G \rangle(\Delta R - \Sigma)}, \tag{18}$$

as

$$\Sigma = \langle \Delta R \langle G \rangle T \rangle = \langle T \langle G \rangle \Delta R \rangle. \tag{19}$$

In this equation, both $\langle G \rangle$ and T depend on the self-energy Σ. However, to lowest order in the fluctuations, T will be given simply by $T = \Delta R$, the fluctuation in the half-space scattering amplitudes. The self-energy will be determined by the Dyson equation

$$\Sigma = \langle \Delta R \frac{1}{1 - R_0(\langle R \rangle + \Sigma)} R_0 \Delta R \rangle. \tag{20}$$

When the conditional (on R_0) mean of R_{net} is expressed in terms of Σ, it becomes

$$\langle R_{net}\rangle|_{R_0} = (\langle R\rangle + \Sigma)\frac{1}{1 - R_0(\langle R\rangle + \Sigma)}. \tag{21}$$

If this result is compared with Eq.15, it can be seen that the mean of R_{net} behaves as if it were the net reflection matrix associated with a flat interface with an effective half-space reflection matrix

$$R_{eff} = \langle R\rangle + \Sigma. \tag{22}$$

In the Bourret approximation, the self-energy is assumed small and is dropped from the right side of Eq. 20,

$$\Sigma = \langle \Delta R\frac{1}{1 - R_0\langle R\rangle}R_0\Delta R\rangle = \langle \Delta R R_0\frac{1}{1 - \langle R\rangle R_0}\Delta R\rangle \tag{23}$$

Even in this Bourret approximation, the self-energy is still random because R_0, which depends on the remainder of the waveguide, is random. However the operator $R_0\frac{1}{1-\langle R\rangle R_0}$ is of the same form as that in equation 15 with $R \rightarrow R_0$ and $R_0 \rightarrow \langle R\rangle$. Its average, now over the fluctuations of R_0, can be written directly as

$$\langle R_0\frac{1}{1 - \langle R\rangle R_0}\rangle = (\langle R_0\rangle + \Sigma_0)\frac{1}{1 - \langle R\rangle(\langle R_0\rangle + \Sigma_0)}. \tag{24}$$

The mean $\langle R_0\rangle$ should be calculated using mean half-space reflection and transmission for the other interfaces in the problem, and the self-energy Σ_0 is given by the Dyson equation

$$\Sigma_0 = \langle \Delta R_0\frac{1}{1 - \langle R\rangle(\langle R_0\rangle + \Sigma_0)}\langle R\rangle\Delta R_0\rangle \approx \langle \Delta R_0\frac{1}{1 - \langle R\rangle\langle R_0\rangle)}\langle R\rangle\Delta R_0\rangle. \tag{25}$$

To this level of approximation, the mean self-energy, averaging over all interfaces, becomes

$$\langle\Sigma\rangle = \langle \Delta R\frac{1}{1 - \langle R_0\rangle\langle R\rangle}\langle R_0\rangle\Delta R\rangle. \tag{26}$$

The effective reflection and transmission matrix is now

$$R_{eff} = \langle R\rangle + \langle\Sigma\rangle. \tag{27}$$

Assuming that roughness on the interfaces is statistically homogeneous implies that R_{eff} is diagonal in wave number. The factor $1 - \langle R_0\rangle\langle R\rangle$ will produce poles corresponding to normal modes. However, since averaged quantities are used here, these poles will be pushed off the real axis.

II.2 An algorithm for the mean field

The mean field in a waveguide with statistically homogeneous rough interfaces can be calculated as follows:

1) Approximate half-space reflection and transmission amplitudes for each interface using c_i and ρ_i above and below the interface. As long as fluctuations are reasonably small, these needn't be perturbative in surface roughness. For example, one could approximate these amplitudes using the lowest order small-slope approximation [3, 6].

2) Calculate the horizontal wavenumber representation of a solution $\psi(\mathbf{r} = (\mathbf{R}, z))$ of the wave equation in the waveguide with flat interfaces

$$\psi(\mathbf{K}, z) = \frac{1}{\sqrt{(2\pi)^d}} \int \exp(-i\mathbf{K} \cdot \mathbf{R})\psi(\mathbf{R}, z)d\mathbf{R}.$$

using boundary conditions implied by the mean half-space reflection and transmission amplitudes. Apparently this is the solution calculated by Kuperman and Schmidt [8]. From this solution one can find the impedance Z just above and just below any interface. For example, for an interface at $z = z_I$,

$$Z(\mathbf{K}, +) = \rho^+\psi(\mathbf{K}, z_I+)/\partial_z\psi(\mathbf{K}, z_I^+). \tag{28}$$

Normally this impedance will not be continuous across the interface. From the impedance Z, the mean reflection coefficients $\langle R_0 \rangle$ are found from

$$\langle R_{0,1}(\mathbf{K}) \rangle = \exp(2i\beta(\mathbf{K}, +)z_I)\frac{i\beta(\mathbf{K}, +)Z(\mathbf{K}, z_I^+)/\rho^+ + 1}{i\beta(\mathbf{K}, +)Z(\mathbf{K}, z_I^+)/\rho^+ - 1} \tag{29}$$

$$\langle R_{0,2}(\mathbf{K}) \rangle = \exp(-2i\beta(\mathbf{K}, -)z_I)\frac{-i\beta(\mathbf{K}, -)Z(\mathbf{K}, z_I^-)/\rho^- + 1}{-i\beta(\mathbf{K}, -)Z(\mathbf{K}, z_I^-)/\rho^- - 1}. \tag{30}$$

Here $\beta(\mathbf{K}, \pm)$ is the vertical component of the wavevector associated with $\mathbf{K}$:

$$\beta(\mathbf{K}, \pm) = \sqrt{(\frac{\omega}{c(\pm)})^2 - \mathbf{K}^2}, \tag{31}$$

where the imaginary and real parts are non-negative.

3) Calculate $\langle \Sigma \rangle$ for each interface using Eq.26 and add to $\langle R \rangle$ calculated to second order in fluctuations to find R_{eff} for each interface (see section III.1 below).

4) Calculate the mean field in the waveguide by calculating the field with flat interfaces, but now with the effective reflection and transmission coefficents given by R_{eff}. Since these effective

reflection and transmission coefficients differ from their flat interface, half-space counterparts, the normal derivatives and the densities times the fields will not be continuous across the effective interfaces. Section III.3 shows how, given a field with sources s satisfying the usual continuity conditions on the flat boundaries between interfaces, one can find a field arising from the same sources but satisfying the jump condition implied by R_{eff}.

A difficulty with this algorthim is that $\langle R_0 \rangle$ needs to be known at each interface as a function of wave number. A computer code like SAFARI [7] provides a numerical solution of the flat interface problem as a function of horizontal wavevector. This solution can be used in Eqs. 23,24 and 25 to find $\langle R_0 \rangle$. $\langle R_0 \rangle$ needs to be known to sufficient resolution that the integral implicit in Eq. 26 can be performed. However, it also appears that a code like SAFARI can be used to find the mean field given the effective reflection and transmission coefficients [8, 9].

III Applications

In this section these ideas will be amplified in discussions of three topics. First, the case of a waveguide with only two interfaces will be considered in order to make contact with the earlier work of Bass and Fuks. Second, a possible iterative solution of the Dyson Equation Eq.20 will be considered to determine when the correction Σ to the mean reflection coefficient is likely to be important. Third, the dispersion relation in a many layered waveguide will be considered.

III.1 Relation to earlier work

In order make contact with the work of Bass and Fuks [10] consider one layer bounded by two rough statistically independent interfaces whose means are at $z = 0$ and $z = -H$. Assume that the sound speed and density are constant between the interfaces. Above and below the layer there may be other layers containing other rough interfaces and variable sound speeds and densities. However suppose that the structure above the upper interface and the interface itself can be characterized by a scattering amplitude $R_U(\mathbf{K}, \mathbf{Q})$ for scattering up-going waves into down-going waves at the upper interface, as in Fig. 3. Likewise let $R_L(\mathbf{K}, \mathbf{Q})$ describe the scattering of down-going waves into up-going waves by the lower interface and everything below it. In this way the interfaces are characterized by single scattering amplitudes rather than a matrix of reflection and transmission amplitudes. This makes the formalism of the preceding sections easier to apply.

To begin, consider only the upper interface and assume the lower interface is flat. Since Bass and Fuks develop a formalism for the impedance, write R_U in terms of a non-local impedance operator Z

$$R_U(\mathbf{K},\mathbf{Q}) = \int (\frac{1}{1+i\eta Z})|_{\mathbf{K},\mathbf{P}}[-1 + i\eta(\mathbf{P})Z(\mathbf{P},\mathbf{Q})]d\mathbf{P}. \tag{32}$$

or more succinctly

$$R_U = (\frac{1}{1+i\eta Z})(-1+i\eta Z) \tag{33}$$

Here $(\frac{1}{1+i\eta Z})|_{\mathbf{K},\mathbf{P}}$ is the $(\mathbf{K},\mathbf{P})$ element of the operator inverse of $1+i\eta Z$ and '1' is understood to indicate the identity operator :

$$1_{\mathbf{K},\mathbf{P}} \equiv \delta(\mathbf{K}-\mathbf{P}).$$

The diagonal operator η is given by

$$\eta(\mathbf{K},\mathbf{P}) \equiv \delta(\mathbf{K}-\mathbf{Q})\beta(\mathbf{K}).$$

Describing scattering in terms of the impedance operator has certain advantages as outlined by Brown et al. For example, if Z is Hermitian, then R is energy conserving.

Write the impedance as the sum of its mean $\langle Z\rangle$ and a fluctuation:

$$Z = \langle Z\rangle + \Delta Z. \tag{34}$$

Then some operator algebra shows that the reflection operator $R = R_U$ can be written

$$R = R_{\langle Z\rangle} + 2\frac{1}{1+\langle Z\rangle i\eta}\Delta Z\frac{1}{1+i\eta\frac{1}{1+\langle Z\rangle i\eta}\Delta Z}i\eta\frac{1}{1+\langle Z\rangle i\eta}, \tag{35}$$

where

$$R_{\langle Z\rangle} = (\frac{1}{1+i\eta\langle Z\rangle})(-1+i\eta\langle Z\rangle). \tag{36}$$

If

$$\mathcal{G} = i\eta(1-R)/2 = i\eta\frac{1}{1+i\eta Z}$$

and

$$G_o = i\eta\frac{1}{1+\langle Z\rangle i\eta},$$

then Equation 35 is recognized to be of the same form for the surface Green function used by Brown et al [11].

Equation 35 also lends itself to expansion in powers of the fluctuations of the impedance, ΔZ. Thus, to first order in ΔZ, R is given by

$$R = R_{\langle Z\rangle} + (2/i\eta)G_o\Delta Z G_o. \tag{37}$$

The first non-vanishing difference between $\langle R\rangle$ and $R_{\langle Z\rangle}$ is second order in ΔZ, since $\langle\Delta Z\rangle = 0$. Thus,

$$\langle R\rangle = R_{\langle Z\rangle} - (2/i\eta)G_o\langle\Delta Z G_o \Delta Z\rangle + \mathcal{O}((\Delta Z)^4). \tag{38}$$

This means that the fluctuation of R about its mean $\langle R\rangle$ is given by

$$\Delta R \equiv R - \langle R\rangle = (2/i\eta)G_o\Delta Z G_o, \tag{39}$$

to first order in the fluctuation of the impedance.

One reason for introducing Z and ΔZ is that now both $\langle R\rangle$ and Σ can be expressed in terms of ΔZ. If Eq.39 is used in Eq.20 and the result combined with Eq.38 , the following result for $R_{eff} = \langle R\rangle + \Sigma$ is obtained:

$$\begin{gathered} R_{eff} = R_{\langle Z\rangle} \\ - (2/i\eta)G_o\langle\Delta Z i\eta(1 - R_{\langle Z\rangle})\frac{1}{1 - R_o R_{eff}}(1 - R_o - R_o(R_{eff} - R_{\langle Z\rangle})\Delta Z\rangle G_o. \end{gathered} \tag{40}$$

If R_{eff} is written in terms of an effective impedance, Z_{eff} by

$$R_{eff} = (\frac{1}{1 + i\eta Z_{eff}})(-1 + i\eta Z_{eff}), \tag{41}$$

then

$$R_{eff} - R_{\langle Z\rangle} \approx (2/i\eta)G_o(Z_{eff} - \langle Z\rangle)G_o. \tag{42}$$

If the difference between $R_{eff} - R_{\langle Z\rangle}$ is neglected on the right side of Eq.40 then the difference between the effective impedance and the averaged impedance can be written

$$Z_{eff} - \langle Z\rangle = -\langle\Delta Z i\eta(1 - R_{eff})\frac{1}{1 - R_o R_{eff}}(1 - R_o)\Delta Z\rangle. \tag{43}$$

The combination $i\eta(1 - R_{eff})\frac{1}{1 - R_o R_{eff}}(1 - R_o)$ can be recognized as the mixed second derivative of the Helmholtz Green function (source $-\delta(z - z')$) within the layer evaluated in the limit of source and receiver approaching the upper surface $z = 0$ from below, when the upper surface is characterized by R_{eff} and the medium below the upper surface is characterized by R_o. That is

$$\partial_z\partial_{z'}G(z, z')|_{z=0^-, z'=0^-} = (i\eta/2)(1 - R_{eff})\frac{1}{1 - R_o R_{eff}}(1 - R_o). \tag{44}$$

Apart from factors of 2 and π, equation 38 is just the Dyson Equation for the impedance which is given in Bass and Fuks [10], Eq. 36.39. To be more precise, the reflection R_o can be replaced by its effective value without doing violence to the approximation, if fluctuations at the lower interface are of the same order as those on the upper interface. Then G refers to the mean Green function.

III.2 Estimating the self-energy

Now return to Eq.20 for the situation just considered when the matrices are in fact 1×1. Furthermore, assume that the medium between the two interfaces in fact has constant sound speed and density and that the botton interface is really a flat Dirichlet surface located at $z = -H$. This means that

$$R_o(\mathbf{K}) = -\exp(i2\beta(\mathbf{K})H). \tag{45}$$

Statistical homogeneity will be assumed so that the self-energy, Σ, is diagonal in wavevector:

$$\Sigma(\mathbf{K}, \mathbf{Q}) = \delta(\mathbf{K} - \mathbf{Q})\Lambda(\mathbf{K}). \tag{46}$$

The correlations in the fluctuations of the scattering amplitude, ΔR can be written as

$$\langle \Delta R(\mathbf{K}, \mathbf{P})\Delta R(\mathbf{P}, \mathbf{Q})\rangle = \delta(\mathbf{K} - \mathbf{Q})W(\mathbf{K}, \mathbf{Q}). \tag{47}$$

The mean relflection amplitude is also diagonal; it's diagonal part will be denoted here by $R(\mathbf{Q})$. Thus in the case of statistical homogeneity, scattering at the upper interface given, the Dyson equation, Eq.20 becomes a scaler integral equation,

$$\Lambda(\mathbf{K}) = \int d\mathbf{Q} W(\mathbf{K}, \mathbf{Q}) \frac{1}{1 - R_o(\mathbf{Q})[R(\mathbf{Q}) + \Lambda(\mathbf{Q})]} R_o(\mathbf{Q}). \tag{48}$$

Consider only the 2-D case in which the horizontal wavevectors $\mathbf{Q}$, $\mathbf{K}$ etc. are one dimensional and use Eq.45 for the reflection from the lower part of the waveguide. Then the self-energy becomes

$$\Lambda(K) = \int dQ W(K, Q) \frac{-1}{1 + \exp(2iqH)[R(Q) + \Lambda(Q)]} \exp(2iqH), \tag{49}$$

where $q = \beta(Q)$. Expansion in powers of the surface roughness shows that for Dirichlet boundary conditions W is given by

$$W(K, Q) = -4\beta(\mathbf{K})\beta(\mathbf{Q})S(\mathbf{K} - \mathbf{Q}),$$

where $S(\mathbf{K})$ is the spectrum of the surface roughness, normalized so that $\int d\mathbf{K}S(\mathbf{K}) = \sigma^2$, the rms roughness. W vanishes as $\mathbf{K}$ and $\mathbf{Q}$ separate. This should be a general feature of W, even when calculated non-perturbatively. This means that if Eq.49 is examined for large K , W in the integrand will force Q to be large. When Q is large the integrand vanishes because

$$\lim_{|Q|\to\infty} \exp(2iqH) = \exp(-2|Q|H) \to 0. \tag{50}$$

Therefore, taking $\Lambda = 0$ would seem to be a reasonable starting point for finding an iterative solution to Eq.49, finding successive iterates from

$$\Lambda_{n+1}(K) = \int dQW(K,Q)\frac{1}{1+\exp(2iqH)[R(Q)+\Lambda_n(Q)]}\exp(2iqH). \tag{51}$$

It is important to note that R is the mean reflection amplitude, not the flat surface reflection. This means, in contrast to the Bourret approximation used by Bass and Fuks for the effective impedance, poles in the integrand of Eq.49 will be moved off the real axis if $|R| < 1$. Poles that occur when R is taken to be the reflection amplitude of a flat surface correspond to normal modes in the waveguide. Because the roughness will cause $|R| < 1$, computations of the iterates in this scheme are more straight forward than in the scheme described by Bass and Fuks for the impedance.

To make an estimate of the likely importance of the corrections implied by Λ, suppose the upper surface is a slightly rough Dirchlet surface, so that

$$\Delta R((K,Q) \approx -2i\beta(K)\hat{h}(K-Q) \tag{52}$$

$$R(Q) \approx -1 + 2\beta(Q)\int dPS(Q-P)\beta(P) \equiv -1 + r(Q) \tag{53}$$

Following the discussion in Bass and Fuks, Ref.[10] p.479-481, the self-energy is likely to be largest when there is a mode at cut-off, i.e. a mode corresponding to $Q = 0$. In this case an estimate of $\Lambda(K=0)$ can be obtained by expanding the denominator of the integrand about $Q = 0$, and evaluating everything else in the integrand at $Q = 0$. Then $\exp(2iqH)$ goes to unity at cut-off and $q \to k_o = \omega/c$. These approximations require that the function W be considerably broader than the Lorentzian function resulting from the denominator. The result of these approximations is

$$\Lambda(0) \approx W(0,0)\int dQ\frac{1}{1+(1-iHQ^2/k_o)(-1+r(0))}. \tag{54}$$

Making the change of variable $Q = x\sqrt{\frac{k_o r(0)}{H(1-r(0))}}$ gives

$$\Lambda(0) \approx W(0,0)\sqrt{\frac{k_0}{H r(0)(1-r(0))}}\int dx\, 1/(1+ix^2). \tag{55}$$

For a Gaussian roughness spectrum with correlation length l

$$W(\mathbf{K},\mathbf{Q}) = 4(\sigma^2 l)\beta(\mathbf{Q})\beta(\mathbf{K})/\sqrt{(2\pi)^d}\exp[-(\mathbf{K}-\mathbf{Q})^2 l^2/2]$$

and

$$r(0) \approx 2k_o^2\sigma^2.$$

This means that the first non-zero iterative correction to the effective reflection coefficient is approximately, (for d = 1)

$$\Lambda(0) \approx 2\sqrt{\pi}\sqrt{\frac{k_o l^2}{H}}\sqrt{\frac{k_o^2\sigma^2}{1-2k_o^2\sigma^2}}\exp[-i\pi/4] \tag{56}$$

The self-energy is important if $\Lambda(0)$ is greater or comparable to $r(0)$ i.e. if

$$\sqrt{k_o l^2/H(k_o\sigma)^2} \geq 1. \tag{57}$$

The factor $\sqrt{k_o l^2/H}$ is the ratio of the correlation length l to the Fresnel zone for propagating across the waveguide, a ratio that occurs in calculations of propagation through volume inhomogeneities.

This estimate and the importance of the self-energy can be checked using a numerical evaluation of Eq.51 for $n = 0$, using $\Lambda_0(K) = 0$ and approximating $\langle R(Q)\rangle \approx -1 + 2\beta(Q)^2\sigma^2$. Letting

$$k_0 = \pi \tag{58}$$

$$H = 4 \tag{59}$$

$$\sigma = .01 \tag{60}$$

$$l = 1, \tag{61}$$

the estimate given in Eq.56 gives

$$\Lambda(0)_{estimated} = 0.070 - i0.070, \tag{62}$$

while numerical evaluation of Eq. 51 for $n = 0$ gives

$$\Lambda(0)_{numerical} = 0.068 - i0.066. \tag{63}$$

Thus for these waveguide and roughness parameters, the estimate in Eq.56 is not too bad. This waveguide is designed so that there are 4 propagating modes with one mode at $Q = 0$. Note that this estimate scales differently than that given in Bass and Fuks for the impedance self-energy.

More important than the quality of this estimate is the fact that for these parameters, the mean of R calculated in a half-space is given by

$$\langle R(0) \rangle \approx -1 + 0.0020,$$

showing that the self-energy resulting from rattling around between the upper and lower boundaries of the waveguide is considerably larger than the correction to the flat surface result (-1) resulting from the roughness in isolation from other boundaries. For other modes the self-energy continues to be larger than $1 + \langle R(K) \rangle$. However, if there is not a mode near cut-off, the self-energy is small, a fact which may justify neglect of Λ in the work of Kuperman and Schmidt [8, 9].

III.3 The dispersion relation in a many-layered waveguide

In this section we return to the problem of finding an approximation to the mean field in a waveguide with many rough interfaces. As indicated above, the mean field can be calculated by replacing each interface by a flat interface and applying the boundary conditions implied by the *effective* reflection and transmission amplitudes at each of these flat interfaces. Since the effective reflection and transmission concern the mean field, neither the mean pressure nor the mean displacement fields are continuous across the mean interfaces. This subsection will show how to find the mean field given the effective properties of the mean interfaces.

Let ψ be a solution of the Helmholtz equation in a stack of layers using the effective reflections and transmissions at each interface:

$$\rho\nabla(1/\rho)\nabla\psi(\mathbf{r}) + (\omega/c(z))^2\psi(\mathbf{r}) = 0. \tag{64}$$

Let ψ_F be a solution of the same equation, but satisfying unperturbed (flat) boundary conditions at the interfaces, so that ψ_F and $(1\rho)\partial_z\psi_F$ are continuous across the interfaces. Suppose first

that the top-most layer of the stack is in fact a semi-infinite homogeneous half-space and that the fields in this layer are described by an incident plane wave and a reflected plane wave:

$$\psi = \exp(-ikz) + R\exp(ikz) \tag{65}$$

$$\psi_F = \exp(-ikz) + R_F\exp(ikz). \tag{66}$$

If the lower interface of the top layer is at $z = H_1$ and Green's theorem is applied to a region bounded above by a plane at $z = \bar{z}$ and below by $z = H_1$, the difference in reflection coef can be shown to be given by

$$2ik(R - R_F)/\rho = \left[\psi_F \frac{1}{\rho}\partial_z\psi - \frac{1}{\rho}\partial_z\psi_F\right]|_{H_1^+}. \tag{67}$$

Here, dependence on horizontal wavenumbers $\mathbf{K}$ is understood and $k = \beta(\mathbf{K})$. If other interfaces, labeled $j = 2...M$, are located at $z = H_j > H_{j=1}$, application of Green's theorem to lower layers gives

$$\left[\psi_F \frac{1}{\rho}\partial_z\psi - \frac{1}{\rho}\partial_z\psi_F\right]|_{H_j^-} = \left[\psi_F \frac{1}{\rho}\partial_z\psi - \frac{1}{\rho}\partial_z\psi_F\right]|_{H_{j+1}^+}. \tag{68}$$

In the bottom-most layer assume there are only downgoing waves so that the quantity on the the right of this equation vanishes. Subtracting the left and adding right sides of this equation to and from the right side of Eq.67, and making use of the continuity of ψ_F and $\frac{1}{\rho}\partial_z\psi_F$, allows one to express the difference in reflection coefficients in terms of the discontinuities of ψ and $\frac{1}{\rho}\partial_z\psi$ across each interface:

$$2ik(R - R_F)/\rho = \sum_{j=1}^{M} \psi_F(H_j^+)JD_j + \frac{1}{\rho(H_j^+)}\partial_z\psi_F(H_j^+)JF_j. \tag{69}$$

Here the discontinuities of ψ are given by

$$JD_j = \frac{1}{\rho(H_j^+)}\partial_z\psi(H_j^+) - \frac{1}{\rho(H_j^-)}\partial_z\psi(H_j^-) \tag{70}$$

$$JF_j = -[\psi(H_j^+) - \psi(H_j^-)]. \tag{71}$$

The jumps JD_j and JF_j can be determined from the matrix of effective reflection and transmission coefficients, R_{eff} at each interface. For example, if ϕ_1 is the field incident from above an interface and ϕ_2 is the field incident from below, then (using R for R_{eff})

$$\psi^+ = (1 + R_{1,1})\phi_1 + R_{1,2}\phi_2 \tag{72}$$

$$\psi^- = R_{2,1}\phi_1 + (1 + R_{2,2})\phi_2. \tag{73}$$

Hence, the jump JF is given by

$$JF = [1 + R_{1,1} - R_{2,1}]\phi_1 + [R_{1,2} - (1 + R_{2,2})]\phi_2. \tag{74}$$

The bracketed expressions on the right vanish for flat, unperturbed interfaces. Therefore these terms depend on the deviation of the effective reflection matrix, δR, from the corresponding flat surface reflection matrix. These deviations are second order in impedance fluctuations. Hence to lowest order in these fluctuations, the incident fields, ϕ_i can be calculated using the unperturbed surfaces. The ϕ_i can now be identified by

$$\phi_1 = (\psi_F^+ - \frac{\rho^+}{ik^+}\frac{1}{\rho^+}\partial_z\psi_F+)/2 \tag{75}$$

$$\phi_2 = (\psi_F^- - \frac{1}{ik^-}\partial_z\psi_F-)/2 = (\psi_F^+ - \frac{\rho^-}{ik^-}\frac{1}{\rho^+}\partial_z\psi_F+)/2 \tag{76}$$

Combining these results gives

$$JF = (N_{1,1}\frac{1}{\rho+}\partial_z\psi^+ + N_{1,2}\psi_F^+)/2 \tag{77}$$

where

$$N_{1,1} = \left[-\frac{\rho^+}{ik^+}(\delta R_{1,1} - \delta R_{2,1}) + \frac{\rho^-}{ik^-}(\delta R_{1,2} - \delta R_{2,2})\right], \tag{78}$$

$$N_{1,2} = [(\delta R_{1,1} - \delta R_{2,1}) + (\delta R_{1,2} - \delta R_{2,2})]. \tag{79}$$

Similarly, the jump in the derivative can be written

$$JD = (N_{2,1}\frac{1}{\rho+}\partial_z\psi^+ + N_{2,2}\psi_F^+)/2, \tag{80}$$

where

$$N_{2,1} = -\left[(-ik^+\delta R_{1,1} - ik^-\delta R_{2,1})\frac{\rho^+}{ik^+} + (ik^+\delta R_{1,2} + ik^-\delta R_{2,2})\frac{\rho^-}{ik^-}\right], \tag{81}$$

$$N_{2,2} = -\left[(ik^+\delta R_{1,1} + ik^-\delta R_{2,1}) + (ik^+\delta R_{2,1} + ik^-\delta R_{2,2})\right]. \tag{82}$$

The point of writing jumps in this way is that now the difference in reflection coefficients at the top of the stack of layers can written succinctly in a matrix form as

$$2ik(R - R_F)/\rho = -(1/2)\sum_{j=1}^{M} V_j^T N^j V_j, \tag{83}$$

where V_j is the column vector

$$V_j = \begin{pmatrix} \frac{1}{\rho(H_j^+)}\partial_z \psi_F(H_j^+) \\ \psi_F(H_j^+) \end{pmatrix}. \tag{84}$$

If instead of a half-space above the stack there is a medium with effective reflection coefficient R_T just above H_1, then the zeros of $1 - R_T R$ determine the horizontal wavenumbers of the normal modes of this waveguide. Suppose the flat interfaced waveguide has a normal mode with wavenumber Q_n, i.e

$$1 - R_T(Q_n)R_F(Q_n) = 0. \tag{85}$$

Also suppose that the right side of Eq.83 is small so that there is a normal mode for the mean field with wave number near Q_n. Writing this wavenumber as $Q_n + q_n$ and expanding the dispersion relation gives

$$q_n \approx -\rho(H_1^+)R_T(Q_n)/[\partial_Q(R_T R_F)|_{Q_n} 4ik(H_1^+)] \sum_{j=1}^{M} V_j^T N^j V_j|_{Q_n}. \tag{86}$$

The imaginary part of q_n gives modal attenuation in the mean field caused by scattering. This generalizes the results of Bass and Fuks by incorporating penetrable surfaces . If the matrix N is computed with mean half-space reflection and transmission amplitudes, then apparently the algorithm of Kuperman and Schmidt is reproduced.

IV Summary

The point of this work has been to show how the mean half-space scattering amplitudes are altered by mechanisms return sound to a scattering surface. These mechanisms, resulting from scattering at additional interfaces or from sound speed profiles which return energy to the scattering surface, act to produce effective reflection and transmission coefficients for the mean field. These effective coefficients are the sum of the mean coefficients calculated when the interface separates two half-spaces, plus a self-energy which accounts for scattering between interfaces and for variable sound speeds or densities. It is necessary to add the self-energy to the mean half-space reflection and transmission coefficients in order to account for the added possibilities for forward scattering. It was shown in section III.2 that the self-energy contribution to the effective reflection coefficient can be significant when there is a mode near cut-off. This is consistent with the results of Bass and Fuks.

Many of the results here have appeared before in the literature, particularly in the book by Bass and Fuks [10] and in the work of Voronovich [5]. What is new here is the possibility of incorporating non-perturbative scattering amplitudes into the description of propagation in a waveguide. Furthermore, the problem of transmission is not treated by Bass and Fuks nor Voronovich.

An algorithm for computing the mean field in a many-layered waveguide was described in section II.2. The ingredients of this algorithm have already been developed separately e.g., code for computing flat interface solutions of the wave equation and approximations for fluctuations of half-space scattering amplitudes. If one is content with the Bourret approximation (Λ_1) for the self-energy, then this can also be computed relatively easily. The effective reflection and transmission coefficients can be used in code similar to that of Kuperman and Schmidt's [8] adaptation of the SAFARI code [7]. It should be noted that even if the self-energy is small, the present formalism still adds the possibility of using non-perturbative half-space reflection and transmission coefficients to the discussion given by Kuperman and Schmidt. Still needed are good ways of computing the self-energy beyond the Bourret approximation, and a way of computing the second moment of fields in a waveguide, using the mean field, including self-energy effects, to drive the scattering, instead of the unperturbed field.

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Figure Captions

Figure 1. To treat variable sound speeds, replace the true profile and density by a sound speed and density that become constant at z_1 and z_2, which can be made aribtrarly close to the mean interface. There will be reflections from the imaginary interfaces at z_1 and z_2 because of the structure of the remainder of the waveguide.

Figure 2. A graphical computation of the net incident field. The fields ϕ_1 and ϕ_2 impinge from above and below the scattering interface. These are reflected and transmitted according to the matrix R and then returned to the interface by $R_{0,1}$ which depends on the structure of the waveguide above the upper dashed line and by $R_{0,2}$ which depends on the structure below the lower dashed line. Here R is a 2×2 matrix operator.

Figure 3. This illustrates the same notion as figure 2. Now, however, R is a scalar operator which describes reflection but not transmission at the upper interface. This operator must account for all the structure above the upper interface. In addition, in this context, the fields ψ and ϕ are scalars.

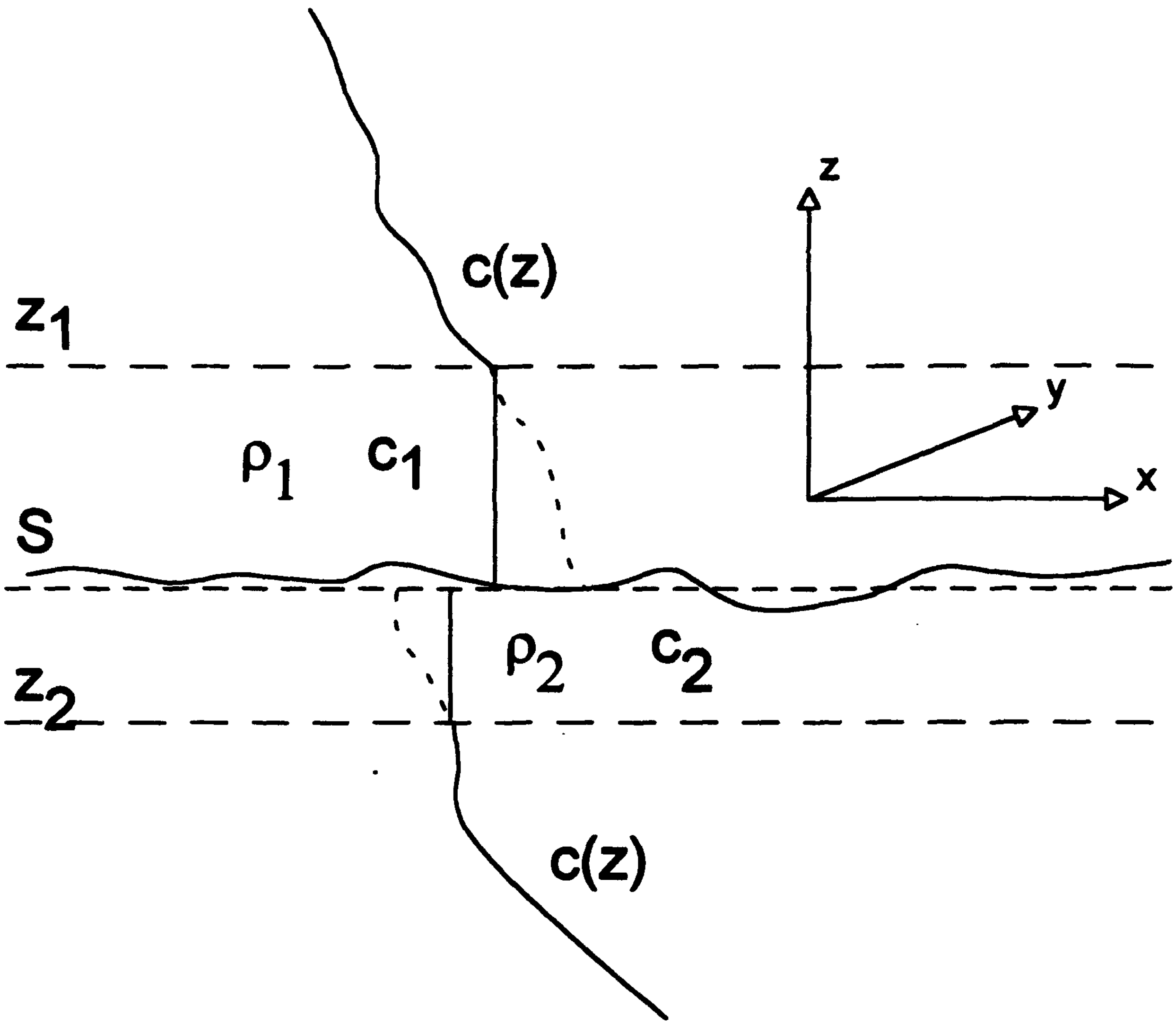

Fig. 1

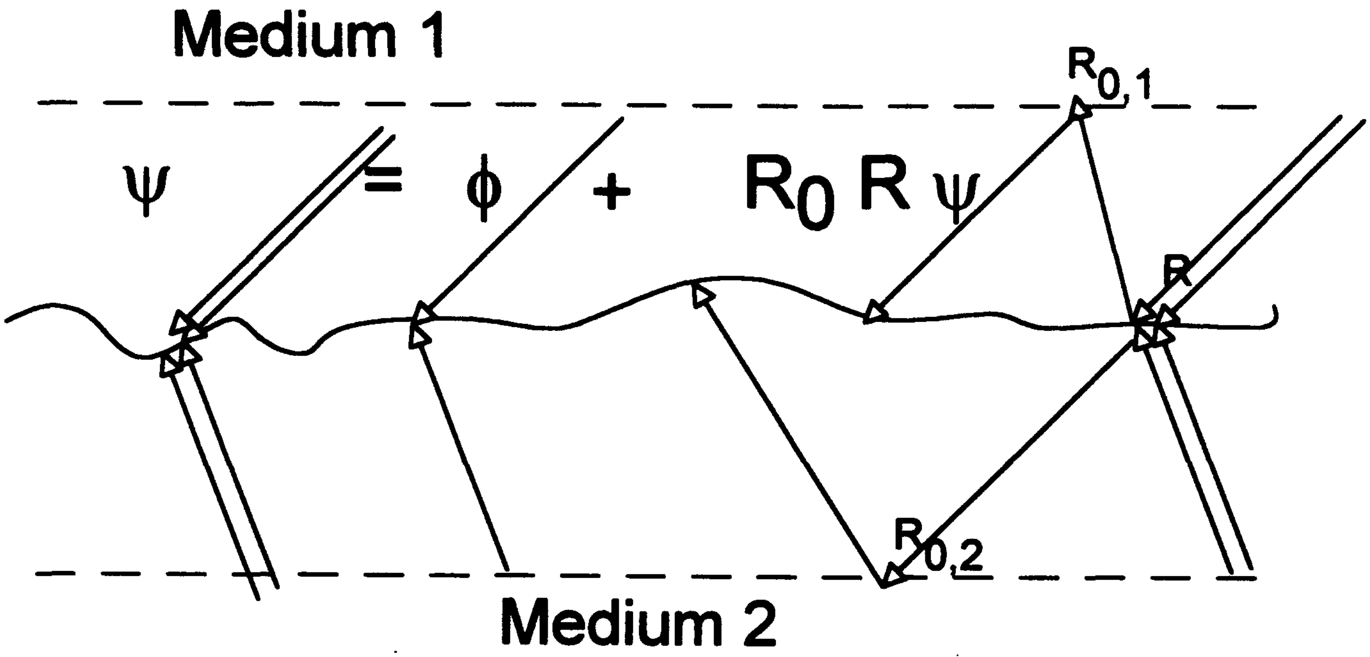

Fig. 2

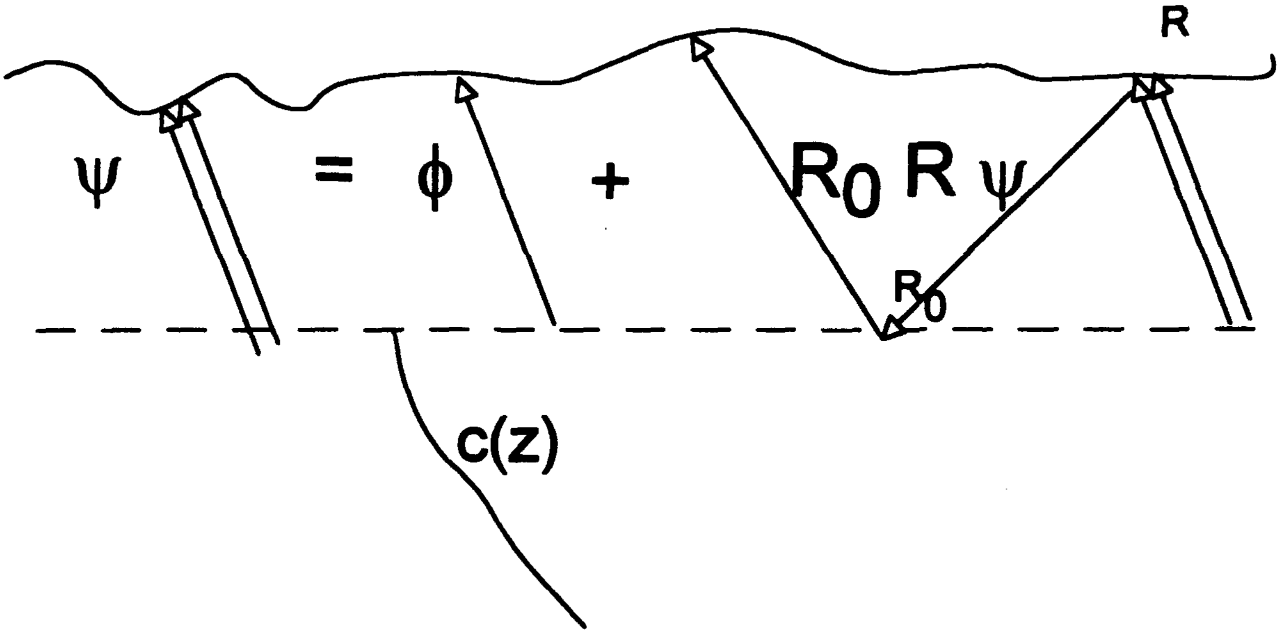

Fig. 3

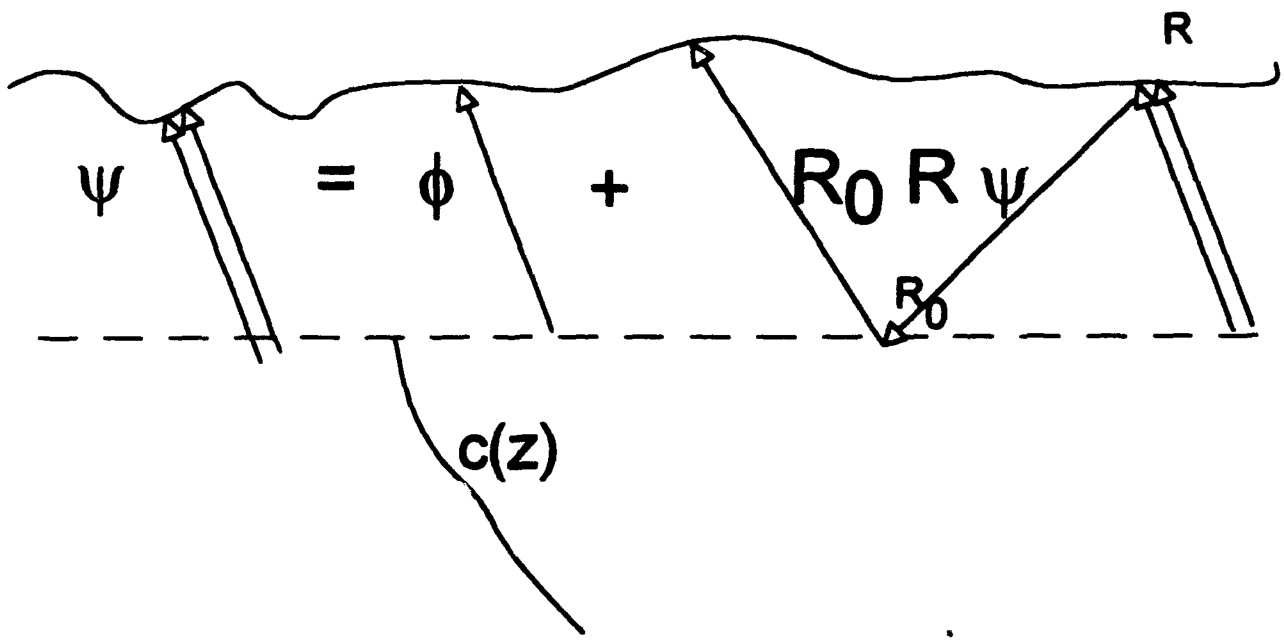

Fig. 3